Linear Einstein equations and Kerr-Schild maps

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Abstract. We prove that given a solution of the Einstein equations g_{ab} for the matter field T_{ab}, an autoparallel null vector field l^a and a solution $(l_a l_c,\ \mathcal{T}_{ac})$ of the linearized Einstein equation on the given background, the Kerr-Schild metric $g_{ac} + \lambda l_a l_c$ (λ arbitrary constant) is an exact solution of the Einstein equation for the energy-momentum tensor $T_{ac} + \lambda \mathcal{T}_{ac} + \lambda^2 l_{(a}\mathcal{T}_{c)b} l^b$. The mixed form of the Einstein equation for Kerr-Schild metrics with autoparallel null congruence is also linear. Some more technical conditions hold when the null congruence is not autoparallel. These results generalize previous theorems for vacuum due to Xanthopoulos and for flat seed space-time due to Gürses and Gürsey.

1. Introduction

In 1978 Xanthopoulos has proved a remarkable theorem [1], establishing the linearity of the Einstein equations for an important class of *vacuum* metrics. According to his result, given an exact solution g of the vacuum Einstein equation $R_{ac}[g] = 0$ and a null vector field l^a, any solution of the linearized vacuum Einstein equation $dR_{ac}[g(\lambda)]/d\lambda\ |_{\lambda=0}= 0$ of the form $dg_{ac}(\lambda)/d\lambda\ |_{\lambda=0}= l_a l_c$ corresponds to the *exact* vacuum solution $g_{ac} + \lambda l_a l_c$ (λ arbitrary, not necessarily small). Such solutions are of Kerr-Schild (KS) type ‡ , initially introduced to find exact solutions starting from a flat seed metric g_{ac} [2, 3]. The theorem proved by Xanthopoulos assures that exact KS type vacuum solutions can be found by solving the linearized vacuum Einstein equation.

A systematic study of this linearized equation for vacuum-vacuum KS maps in a Newman-Penrose framework revealed [4, 5] that the generic case with the congruence l^a shearing does not contain the shear-free case as a smooth limit. For flat seed space-time the solution of the KS problem is contained in the Kerr theorem [6, 7], which gives all shear-free geodesic null congruences l^a explicitly. The same theorem provides valuable help in the search of KS space-times with conformally flat seed space-times [8]-[11]. The generic shearing case for vacuum was solved both for twisting [12] and nontwisting null congruences [13].

An other result relating the linearity of the Einstein equation to KS maps was established by Gürses and Gürsey [14]. They have shown the linearity of the *mixed*

‡ Kerr-Schild metrics are frequently considered in the form $\tilde{g}_{ab} = g_{ab} + V\hat{l}_a\hat{l}_b$, with V an arbitrary function. Then a *geodesic curve* with tangent $\hat{l}^a$ can be affinely parametrized such that $\hat{l}^b\nabla_b\hat{l}^a = 0$ and the curve is actually a *geodesic*. We prefer to reparametrize the curve so that $l_a l_b$ absorbes the potential V, modulo an arbitrary constant λ. The price to pay is that affine parametrization cannot be required for geodesic curves, therefore we call the tangents to geodesic curves autoparallel vector fields which obey $l^b\nabla_b l^a = f l^a$.

form of Einstein equations for generic KS metrics with *flat* seed space-time and *geodesic* KS congruence. Their proof is based on the remark that for KS space-times with flat seed metric a Lorentz-covariant coordinate system can be chosen for which both the Einstein and Landau energy-momentum pseudotensors vanish. Their result is in a sense more restrictive (flat seed metric, geodesic KS congruence), in other sense more general (the KS metric not necessarily vacuum) than the Xanthopoulos theorem. The linearity proved by Gürses and Gürsey however holds only for the mixed form of the Einstein equation.

The natural question arises whether these two results can be generalized? In this paper first we find a theorem similar to the one proved by Xanthopoulos holding in the presence of matter. Basically we establish the set of conditions to impose on the difference of the seed and KS energy-momentum tensors which assure the linearity of the Einstein equation §. For this purpose, in section 2 we derive the KS equations, which split into a homogeneous and an inhomogeneous subsystem. With the matter source of the seed metric left arbitrary we seek the KS energy-momentum tensor in the form of a sum of the seed energy-momentum tensor and a polynomial in λ. In the vacuum-vacuum subcase the result of Xanthopoulos is easily recovered.

In section 3 we solve the homogeneous equations, then we turn to the analysis of the inhomogeneous subsystem, composed of the third, second and first order equations in λ. Then we replace the third and second order equations with appropiate conditions on the energy-momentum tensor. These prove our Theorem 1.

Section 4 deals with the particular case of autoparallel congruences. The main result of the paper is announced as Theorem 2. Then some consequences of the linear equation are exploited in order to further simplify the KS energy-momentum tensor.

Finally in section 5 we study the mixed form of the Einstein equation for KS metrics. In case of generic congruences we find that the Einstein equation is quadratic. A *sufficient condition* assuring linearity is the autoparallelism of the congruence. The result of Gürses and Gürsey arises therefore as a special case. We establish and generalize their result in a covariant manner. Their restriction to flat seed space-time is unnecesary.

2. Generalized KS maps

Two connections ∇ and $\tilde{\nabla}$ annihilating the metrics g_{ac} and $\tilde{g}_{ac}$ are related [17] as:

$$\tilde{\nabla}_a\omega_b - \nabla_a\omega_b = C^c{}_{ab}\omega_c \tag{1}$$

where ω_a is any one-form and $C^c{}_{ab}$ is given by

$$C^c{}_{ab} = \frac{1}{2}\tilde{g}^{cd}(\nabla_a\tilde{g}_{bd} + \nabla_b\tilde{g}_{ad} - \nabla_d\tilde{g}_{ab}) \ . \tag{2}$$

The relation between the Ricci tensors is straightforward:

$$\tilde{R}_{ac} = R_{ac} - 2\nabla_{[a}C^b{}_{b]c} + 2C^e{}_{c[a}C^b{}_{b]e} \ . \tag{3}$$

When the two metrics are the seed metric and the KS metric of the KS map, defined as

$$\tilde{g}_{ac} = g_{ac} + \lambda l_a l_c \ , \qquad \tilde{g}^{ac} = g^{ac} - \lambda l^a l^c \ , \qquad l_a l^a = 0 \ , \tag{4}$$

§ Our approach differs from previous generalizations allowing the same seed and KS energy-momentum tensors which rely on imposition of matter-specific constraints on the null KS vector [15]-[16].

(λ arbitrary parameter), Eq. (3) takes the detailed form:

$$\tilde{R}_{ac} = R_{ac} + \lambda R^{(1)}_{ac} + \lambda^2 R^{(2)}_{ac} + \lambda^3 R^{(3)}_{ac} \tag{5}$$

with the coefficients $R^{(i)}_{ac}$ given by

$$R^{(1)}_{ac} = \nabla_b[\nabla_{(a}(l_{c)}l^b) - \frac{1}{2}\nabla^b(l_a l_c)] \ , \tag{6}$$

$$\begin{aligned} R^{(2)}_{ac} = & \nabla_b l^b l_{(a} D l_{c)} + \frac{1}{2} D l_a D l_c + l_{(a} D D l_{c)} \\ & + l_a l_c \nabla_b l_d \nabla^{[b} l^{d]} - D l^b \nabla_b l_{(a} l_{c)} \ , \end{aligned} \tag{7}$$

$$R^{(3)}_{ac} = -\frac{1}{2} l_a l_c D l^b D l_b \ . \tag{8}$$

(We have denoted $D = l^b \nabla_b$. Round and square brackets on index pairs denote symmetrization and antisymmetrization, respectively.) For later convenience we enlist the relations:

$$R^{(1)}_{ac} g^{ac} = (\nabla_b D + D\nabla_b) l^b + (\nabla_b l^b)^2 \ , \tag{9}$$

$$\begin{aligned} R^{(1)}_{ac} l^c = & \frac{1}{2}[-D l^b \nabla_a l_b + \nabla_b l^b D l_a + D D l_a \\ & + l_a (2\nabla_b l_d \nabla^{[b} l^{d]} + \nabla_b D l^b)] \ , \end{aligned} \tag{10}$$

$$R^{(1)}_{ac} l^a l^c = -D l^b D l_b \ , \tag{11}$$

$$R^{(2)}_{ac} g^{ac} = -\frac{1}{2} D l^b D l_b \ , \tag{12}$$

$$R^{(2)}_{ac} l^c = -\frac{1}{2} l_a D l^b D l_b \ , \tag{13}$$

$$R^{(3)}_{ac} l^c = 0 = R^{(3)}_{ac} g^{ac} \ . \tag{14}$$

The Einstein equations for the seed and KS space-times

$$R_{ac} = k(T_{ac} - \frac{1}{2} g_{ac} T) \ , \qquad \tilde{R}_{ac} = k(\tilde{T}_{ac} - \frac{1}{2}\tilde{g}_{ac}\tilde{T}) \tag{15}$$

(where $T = T_{bd} g^{bd}$ and $\tilde{T} = \tilde{T}_{bd}\tilde{g}^{bd}$), inserted into Eq. (5) give the KS equation

$$k(\tilde{T}_{ac} - T_{ac}) - \frac{k}{2} g_{ac}(\tilde{T} - T) - \frac{k}{2}\lambda l_a l_c \tilde{T} = \lambda R^{(1)}_{ac} + \lambda^2 R^{(2)}_{ac} + \lambda^3 R^{(3)}_{ac} \ . \tag{16}$$

We search for the difference of the seed and KS energy-momentum tensors in the form of a polynomial in λ

$$\tilde{T}_{ac} = T_{ac} + \lambda \mathcal{T}_{ac} + \lambda^2 \sigma_{ac} + \lambda^3 \pi_{ac} + \sum_{n=1}^{p} \lambda^{3+n} \pi^{(n)}_{ac} \ . \tag{17}$$

(The limit $p \to \infty$ is allowed.) The parameter λ being arbitrary, the KS equation (16) decouples in several equations corresponding to different powers of λ. They form a homogeneous

$$(\lambda^{5+p}\frac{k}{2} l_a l_c) \ : \qquad \pi^{(p)}_{bd} l^b l^d = 0 \ , \tag{18}$$

$$(\lambda^{4+p}\frac{k}{2}) \ : \qquad g_{ac}(\pi^{(p)}_{bd} l^b l^d) + l_a l_c (\pi^{(p-1)}_{bd} l^b l^d - \pi^{(p)}) = 0 \ , \tag{19}$$

$$\begin{aligned} (\lambda^{3+n} k \ , n = \overline{1,p}) \ : \qquad & \pi^{(n)}_{ac} + \frac{1}{2} g_{ac}(\pi^{(n-1)}_{bd} l^b l^d - \pi^{(n)}) \\ & + \frac{1}{2} l_a l_c (\pi^{(n-2)}_{bd} l^b l^d - \pi^{(n-1)}) = 0 \ , \end{aligned} \tag{20}$$

and an inhomogeneous system

$$(\lambda^3)\ :\quad k\left[\pi_{ac}+\frac{1}{2}g_{ac}(\sigma_{bd}l^bl^d-\pi)+\frac{1}{2}l_al_c(\mathcal{T}_{bd}l^bl^d-\sigma)\right]=R^{(3)}_{ac}\ , \tag{21}$$

$$(\lambda^2)\ :\quad k\left[\sigma_{ac}+\frac{1}{2}g_{ac}(\mathcal{T}_{bd}l^bl^d-\sigma)+\frac{1}{2}l_al_c(T_{bd}l^bl^d-\mathcal{T})\right]=R^{(2)}_{ac}\ , \tag{22}$$

$$(\lambda)\quad :\quad k\left[\mathcal{T}_{ac}+\frac{1}{2}g_{ac}(T_{bd}l^bl^d-\mathcal{T})-\frac{1}{2}l_al_cT\right]=R^{(1)}_{ac}\ . \tag{23}$$

The brackets preceeding each equation contain the corresponding powers of λ together with the nonessential multiplying factors, which were dropped. In Eq. (20) $\pi^{(0)}_{ac}=\pi_{ac}$ and $\pi^{(-1)}_{ac}=\sigma_{ac}$. The traces $\tau,\ \sigma,\ \pi$ and $\pi^{(n)}$ are formed with the metric g.

Eq. (23) is obviously the linearized Einstein equation. It can be obtained independently by the standard linearization procedure, seeking for approximate solutions $g_{ac}(\lambda)=g_{ac}+\lambda l_al_c$ in the presence of the energy-momentum tensor $T_{ac}(\lambda)=T_{ac}+\lambda\mathcal{T}_{ac}$ (with λ small). Our aim is to show under what circumstances Eq. (23) implies the rest of the KS equations (18)-(22).

Vacuum subcase For vacuum-vacuum KS maps all coefficients in the expansion (17) vanish, thus Eqs. (18)-(20) are trivial and Eqs. (21)-(23) reduce to $R^{(i)}_{ac}=0,\ i=\overline{1,3}$. The twice contracted equation $R^{(1)}_{ac}l^al^c=0$ by virtue of Eq. (11) carries the same information as $R^{(3)}_{ac}=0$, namely $Dl^b=fl^b$. Inserting the condition of autoparallelism in $R^{(i)}_{ac}=0,\ i=1,2$ by virtue of Eq. (10) we find

$$0=R^{(2)}_{ac}=l_al_c\left[Df+\frac{f^2}{2}+f\nabla_bl^b+\nabla_bl_d\nabla^{[b}l^{d]}\right]\ , \tag{24}$$

$$0=R^{(1)}_{ac}l^c=l_a\left[Df+\frac{f^2}{2}+f\nabla_bl^b+\nabla_bl_d\nabla^{[b}l^{d]}\right]\ . \tag{25}$$

Thus $R^{(2)}_{ac}=0$ is consequence of $R^{(1)}_{ac}=0$ and all the information is encoded in the latter equation. Recalling that $R^{(1)}_{ac}=0$ is the linearized Einstein equation, we have recovered the result of Xanthopoulos.

3. Solution of the nonlinear KS equations

Next we study the generic system (18)-(23).

The Homogenous Subsystem The contraction of Eq. (20) with l^al^c gives $\pi^{(n)}_{ac}l^al^c=0$. Then from the trace of Eq. (20) we find $\pi^{(1)}=2\pi_{bd}l^bl^d$ and $\pi^{(m)}=0,\ m=\overline{2,p}$. Thus Eq. (20) gives the following solution:

$$\pi^{(1)}_{ac}=\frac{1}{2}g_{ac}(\pi_{bd}l^bl^d)-\frac{1}{2}l_al_c(\sigma_{bd}l^bl^d-\pi) \tag{26}$$

$$\pi^{(2)}_{ac}=\frac{1}{2}l_al_c(\pi_{bd}l^bl^d) \tag{27}$$

$$\pi^{(k)}_{ac}=0\ ,\qquad k=\overline{3,p} \tag{28}$$

Eq. (28) solves the rest of the homogeneous subsystem, Eqs. (18) and (19) too.

The Inhomogeneous Subsystem The contraction of Eq. (23) with l^al^c gives

$$k\mathcal{T}_{ac}l^al^c=-Dl^bDl_b\ . \tag{29}$$

Then the trace and the l^c projection of Eq. (22) imply

$$k\sigma=(-3/2)Dl^bDl_b\ ,\qquad k\sigma_{ac}l^c=(-3/4)l_aDl^bDl_b\ . \tag{30}$$

Inserting these into the third order equation (21), we find:

$$k\pi_{ac} = -\frac{3}{4}l_a l_c Dl^b Dl_b \ . \tag{31}$$

As a by-product we find $\pi^{(1)}_{ac} = 0 = \pi^{(2)}_{ac}$. Therefore *whenever the KS energy-momentum tensor has the form*

$$\tilde{T}_{ac} = T_{ac} + \lambda\mathcal{T}_{ac} + \lambda^2\sigma_{ac} - \frac{3}{4k}\lambda^3 l_a l_c Dl^b Dl_b \ , \tag{32}$$

only the first and second order KS equations are independent.

Next we remark that the l^c projection of the first order equation (23):

$$k\left[\mathcal{T}_{ab}l^b + \frac{1}{2}l_a(T_{bd}l^b l^d - \mathcal{T})\right] = R^{(1)}_{ab}l^b \tag{33}$$

simplifies the second order KS equation (22). We obtain:

$$k\sigma_{ac} = kl_{(a}\mathcal{T}_{c)b}l^b - \frac{1}{4}g_{ac}Dl^b Dl_b + P^{(2)}_{(ac)} \ , \tag{34}$$

where

$$P^{(2)}_{ac} = R^{(2)}_{ac} - l_c R^{(1)}_{ab}l^b \ . \tag{35}$$

From Eqs. (7) and (10) we find:

$$\begin{aligned} 2P^{(2)}_{ac} = {} & \nabla_b l^b l_a Dl_c + Dl_a Dl_c + l_a DDl_c \\ & + Dl^b\left(l_c\nabla_a l_b - 2l_{(a}\nabla_{|b|}l_{c)}\right) - l_a l_c \nabla_b Dl^b \ . \end{aligned} \tag{36}$$

Thus Eq. (34), expressing σ_{ac} in terms of $\mathcal{T}_{ac}$, g_{ac} and l^a is a substitute for the second order equation (22). *Whenever the KS energy-momentum tensor has the expression* (32) *and Eqs.* (34) *and* (36) *hold, the only independent KS equation is the linear one.* We have proved:

Theorem 1 *Let g_{ac} be a solution of the Einstein equations for the matter field T_{ac} and l^a a null vector field. Any solution $(l_a l_c,\ \mathcal{T}_{ac})$ of the linearized Einstein equation on the background $(g_{ac},\ T_{ac})$ corresponds to an exact solution of Kerr-Schild type $g_{ac} + \lambda l_a l_c$ (with λ arbitrary) of the Einstein equation with the energy-momentum tensor given by Eqs. (32), (34) and (36).*

4. Autoparallel Congruences

A more elegant result holds for autoparallel congruences ($Dl^b = fl^b$). A simple check shows

$$P^{(2)}_{ac} = 0 \ , \tag{37}$$

and the content of Eqs. (32) and (34) reduces to

$$\tilde{T}_{ac} = T_{ac} + \lambda\mathcal{T}_{ac} + \lambda^2 l_{(a}\mathcal{T}_{c)b}l^b \ . \tag{38}$$

We announce

Theorem 2 *Let g_{ac} be a solution of the Einstein equations for the matter field T_{ac} and l^a an autoparallel null vector field. Any solution $(l_a l_c,\ \mathcal{T}_{ac})$ of the linearized Einstein equation on the background $(g_{ac},\ T_{ac})$ corresponds to an exact solution of Kerr-Schild type $g_{ac} + \lambda l_a l_c$ (with λ arbitrary) of the Einstein equation with the energy-momentum tensor $T_{ac} + \lambda\mathcal{T}_{ac} + \lambda^2 l_{(a}\mathcal{T}_{c)b}l^b$.*

Some consequences of the remaining linear equation Eq. (23) are immediate when the congruence is autoparallel.

Theorem 3 *The null vector of the KS metric is autoparallel if and only if either of the following conditions hold:*

$$\mathcal{T}_{ac}l^a l^c = 0 \Leftrightarrow (\tilde{T}_{ac} - T_{ac})l^a l^c = 0 \ . \tag{39}$$

Proof. The first statement is obvious from Eq. (29). Then $Dl^b = fl^b$ and (38) give the second statement. ■

Conversely, Eqs. (38) and (39) imply the autoparallelism of the null congruence. Our Theorem 3 generalizes Theorem 28.1 of Kramer *et al.* [18], given there for flat seed space-time.

Theorem 4 *The autoparallel null vector of the KS metric is an eigenvector of the difference of KS and seed energy-momentum tensors.*

$$\mathcal{T}_{ac}l^c = \alpha l_a \Leftrightarrow (\tilde{T}_{ac} - T_{ac})l^c = \alpha l_a \ . \tag{40}$$

Proof. For autoparallel KS congruence the l^c projection of the linear equation, Eq. (33) becomes

$$k\left[\mathcal{T}_{ac}l^c + \frac{1}{2}l_a(T_{bd}l^b l^d - \mathcal{T})\right] = l_a\left(Df + \frac{f^2}{2} + f\nabla_b l^b + \nabla_b l_d \nabla^{[b}l^{d]}\right) \ , \tag{41}$$

which implies the first statement. Then the second statement is consequence of Eq. (38) ■

Eq. (40), first derived by Martin and Senovilla [8] was exploited in the search of KS type perfect fluid solutions [8]-[11]. The restriction of Theorem 4 to flat seed metric is Theorem 28.2 of Kramer *et al.* [18], interpreted as a constitutive constraint for elastic solid type KS solutions [19, 20]. Due to Eq. (40) the energy-momentum tensor (38) becomes

$$\tilde{T}_{ac} = T_{ac} + \lambda \mathcal{T}_{ac} + \lambda^2 \alpha l_a l_c \ . \tag{42}$$

As consequence, the mixed form of the KS energy-momentum tensor is *linear* in λ:

$$\begin{aligned}\tilde{T}_a{}^b = \tilde{T}_{ac}\tilde{g}^{cb} &= (g^{cb} - \lambda l^c l^b)(T_{ac} + \lambda \mathcal{T}_{ac} + \lambda^2 \alpha l_a l_c) \\ &= T_a{}^b + \lambda(\mathcal{T}_a{}^b - l^b T_{ac}l^c) \ .\end{aligned} \tag{43}$$

We compute the undetermined coefficient α as follows. Eqs. (41) and (40) imply

$$k\alpha = Df + \frac{f^2}{2} + f\nabla_b l^b + \nabla_b l_d \nabla^{[b}l^{d]} - \frac{k}{2}(T_{bd}l^b l^d - \mathcal{T}) \ . \tag{44}$$

The trace of the linear KS equation (23) gives

$$k(3T_{bd}l^b l^d - \mathcal{T}) = 2Df + 2f\nabla_b l^b - \nabla_b l_d \nabla^d l^b + (\nabla_b l^b)^2 \ . \tag{45}$$

In the derivation of Eq. (45) we have employed the identity

$$D\nabla_b l^b = \nabla_b D l^b - \nabla_b l_d \nabla^d l^b - R_{bd}l^b l^d \ , \tag{46}$$

the autoparallelism of l^c and the Einstein equations (15). By combining Eqs. (44) and (45) we express α in terms of $f,\ T_{ac},\ g_{ac}$ and l^a:

$$2k\alpha = f^2 + \nabla_b l_d \nabla^b l^d - (\nabla_b l^b)^2 + 2kT_{bd}l^b l^d \ . \tag{47}$$

Then in the announcement of Theorem 2 one can replace the expression (38) with the set (42) and (47).

5. The linearity of the Einstein equations in the mixed form

We would like to see, how the previous results relate to the theorem of Gürses and Gürsey, according to which for flat seed space-time the mixed form of the Einstein equation is linear in λ.

We study generic KS congruences. From the KS condition on the metric (4) we find $\tilde{g}_a^b = g_a^b = \delta_a^b$. From Eq. (3) the following relation between the mixed forms of the Ricci tensors emerges

$$\tilde{R}_a^b = R_a^b + \lambda P_a^{(1)b} + \lambda^2 P_a^{(2)b} + \lambda^3 P_a^{(3)b} + \lambda^4 P_a^{(4)b} \ , \tag{48}$$

with the coefficients $P_a^{(i)b}$ on the right hand side given by

$$P_a^{(1)b} = R_a^{(1)b} - l^b R_{ac} l^c \ , \tag{49}$$
$$P_a^{(2)b} = R_a^{(2)b} - l^b R_{ac}^{(1)} l^c \ , \tag{50}$$
$$P_a^{(3)b} = R_a^{(3)b} - l^b R_{ac}^{(2)} l^c \ , \tag{51}$$
$$P_a^{(4)b} = -\, l^b R_{ac}^{(3)} l^c \ . \tag{52}$$

We have denoted $R_a^{(i)b} = R_{ac}^{(i)} g^{cb}$. From Eqs. (8), (13) and (14) we see immediately that $P_a^{(3)b} = P_a^{(4)b} = 0$, therefore $\tilde{R}_a^b$ is only quadratic in λ. By inserting the mixed form of the Einstein equations (15) into Eq. (48), we obtain the mixed form of the KS equation:

$$k(\tilde{T}_a{}^b - T_a{}^b) - \frac{k}{2}\delta_a{}^b(\tilde{T} - T) = \lambda P_a^{(1)b} + \lambda^2 P_a^{(2)b} \, , \tag{53}$$

or after the elimination of traces

$$k(\tilde{T}_a{}^b - T_a{}^b) = \lambda \left[P_a^{(1)b} - \frac{1}{2}\delta_a{}^b P^{(1)} \right] + \lambda^2 \left[P_a^{(2)b} - \frac{1}{2}\delta_a{}^b P^{(2)} \right] \ . \tag{54}$$

It is obvious that the difference of the mixed forms of the seed and KS energy-momentum tensors is quadratic in λ. Let us denote then

$$\tilde{T}_a{}^b = T_a{}^b + \lambda \mathcal{S}_a^b + \lambda^2 \mathcal{S}_a^{(2)b} \ . \tag{55}$$

Decomposition of Eq. (53) with respect to powers of λ gives:

$$k\left(\mathcal{S}_a^b - \frac{1}{2}\delta_a{}^b \mathcal{S}\right) = P_a^{(1)b} \, . \tag{56}$$

$$k\left(\mathcal{S}_a^{(2)b} - \frac{1}{2}\delta_a{}^b \mathcal{S}^{(2)}\right) = P_a^{(2)b} \, . \tag{57}$$

As before, the contraction of Eq. (56) with l_b results in an equation with the same r.h.s. as Eq. (57):

$$R_a^{(2)b} - k\left(l^b \mathcal{S}_a^c l_c - \frac{1}{2} l_a l^b \mathcal{S}\right) = P_a^{(2)b} \, . \tag{58}$$

Comparison gives the *condition, under which the second order equation is consequence of the first order one*:

$$k\left(\mathcal{S}_a^{(2)b} - \frac{1}{2}\delta_a{}^b \mathcal{S}^{(2)}\right) = R_a^{(2)b} - k\left(l^b \mathcal{S}_a^c l_c - \frac{1}{2} l_a l^b \mathcal{S}\right) \ . \tag{59}$$

Thus we have the following

Corollary 5 *Let g_{ac} be a solution of the Einstein equations for the matter field T_{ac} and l^a a null vector field. Any solution $(l_a l^b,\ \mathcal{S}_a{}^b)$ of the linearized Einstein equation (in mixed form) on the given background corresponds to an exact solution $\delta_a{}^b + \lambda l_a l^b$ (with λ arbitrary) of the (mixed form) Einstein equation with the energy-momentum tensor given by Eqs (55) and (59).*

For autoparallel null congruences $P_a^{(2)b} = P_{ac}^{(2)} g^{cb}$ vanishes by virtue of Eq. (37). Therefore the mixed form of the Einstein equation Eq. (53) becomes linear in λ.

Corollary 6 *Let g_{ac} be a solution of the Einstein equations for the matter field T_{ac} and l^a an autoparallel null vector field. Then the mixed form of the Einstein equation for the KS metric $\delta_a{}^b + \lambda l_a l^b$ and energy-momentum tensor $\tilde{T}_a{}^b = T_a{}^b + \lambda \mathcal{S}_a^b$ is linear in λ.*

Here $\mathcal{S}_a^b = \mathcal{T}_a{}^b - l^b T_{ac} l^c$, cf. Eq. (43). The result of Gürses and Gürsey, established for flat seed metric, is contained as a special case.

6. Concluding Remarks

The main result of the paper is the generalization of the Xanthopoulos theorem. We have found the form of the KS energy-momentum tensor for which the Einstein equations are linear, regardless of the seed energy-momentum tensor. The importance of both versions of our theorem is given by the expectation that solving the linearized Einstein equation is easier (although not trivial at all, as shown by previous analysis of the vacuum case [5],[12] and [13]). Still, in the case of autoparallel null congruences the simplicity of the KS energy-momentum tensor linearizing the Einstein equation is promising.

The second Corollary generalizes the result of Gürses and Gürsey [14] for non-flat seed metrics. Our result relies only on the autoparallelism of the KS congruence. The equivalent condition on the energy-momentum tensors is given by Eq. (39).

Finally we remark that additional constraints on the energy-momentum tensors possibly emerge, when in order to assure that the local energy density measured by any local observer is positive, we impose the weak energy condition on both energy-momentum tensors T_{ac} and $\tilde{T}_{ac}$:

$$T_{ac} t^a t^b \geq 0 \ , \qquad \tilde{T}_{ac} \tilde{t}^a \tilde{t}^b \geq 0 \ . \tag{60}$$

Here t^a and $\tilde{t}^a$ are arbitrary time-like vector fields in the metrics g_{ac} and $\tilde{g}_{ac}$, respectively. These conditions are not immediately exploitable in the general case. In the particular case of algebraically special vacuum seed space-time and autoparallel null congruence for instance Nahmad-Achar [21] has obtained from Eq. (60) a (first order differential) inequality on the spin coefficients.

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